ECS455: Chapter 5

OFDM

5.4 Cyclic Prefix (CP)

Office Hours:
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Wednesday 15:30-16:30
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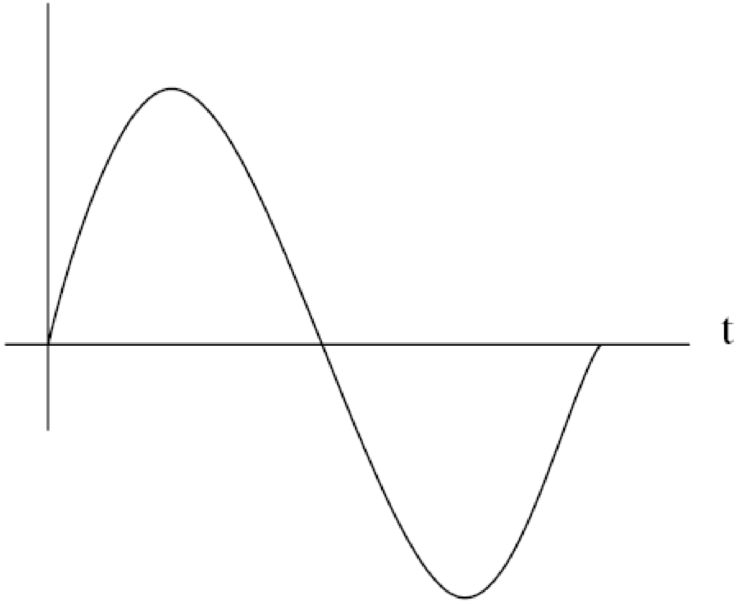
Three steps towards modern OFDM

1. Mitigate Multipath (ISI): Decrease the rate of the original data stream via multicarrier modulation (FDM)
2. Gain Spectral Efficiency: Utilize orthogonality
3. Achieve Efficient Implementation: FFT and IFFT

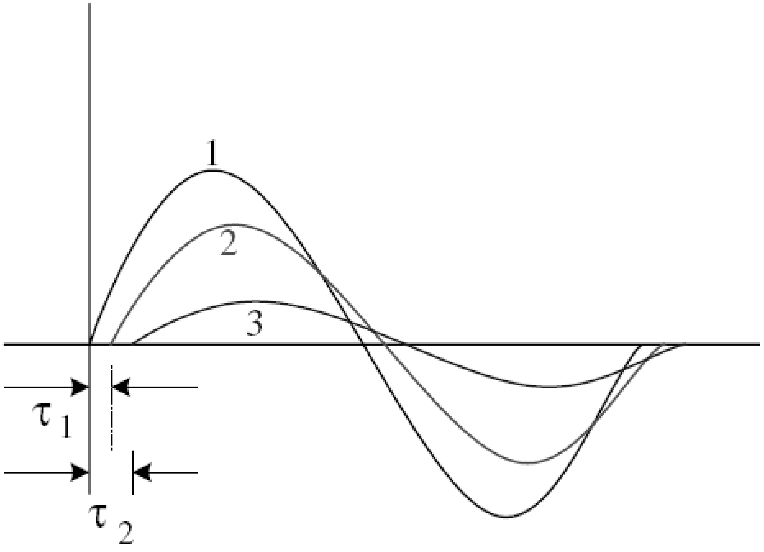
- Extra step: Completely eliminate ISI and ICI
    - Cyclic prefix

Cyclic Prefix: Motivation (1)

- Recall: Multipath Fading and Delay Spread



Transmitted Signal

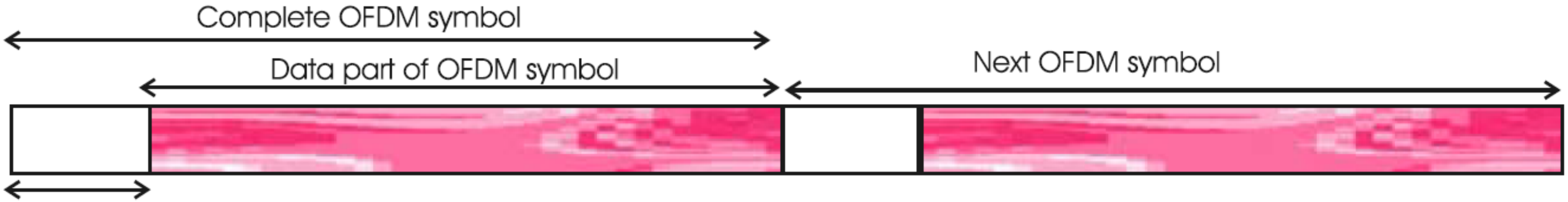


Received Signal

Cyclic Prefix: Motivation (2)

- OFDM uses large symbol duration T_s
  - compared to the duration of the impulse response $\tau_{\max}$ of the channel
  - to reduce the amount of ISI
- **Q**: Can we "eliminate" the multipath (**ISI**) problem?
- **A**: To reduce the ISI, add **guard interval** larger than that of the estimated delay spread.
- If the guard interval is left empty, the orthogonality of the sub-carriers no longer holds, i.e., **ICI** (inter-channel interference) still exists.
- **Solution**: To prevent **both** the **ISI** as well as the **ICI**, OFDM symbol is **cyclically extended** into the guard interval.

Cyclic Prefix

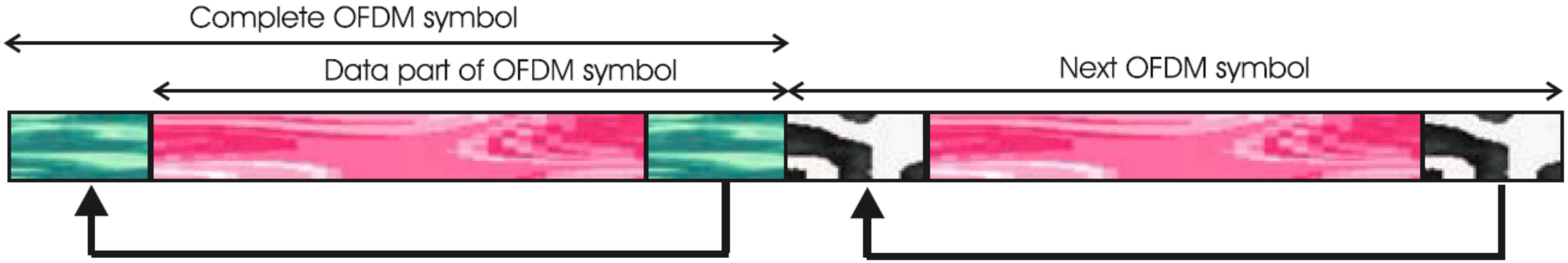


Complete OFDM symbol

Data part of OFDM symbol

Next OFDM symbol

Guard Interval, $T_{CP} > \tau_{max}$
Using empty spaces as guard
interval at the beginning of
each symbol



Complete OFDM symbol

Data part of OFDM symbol

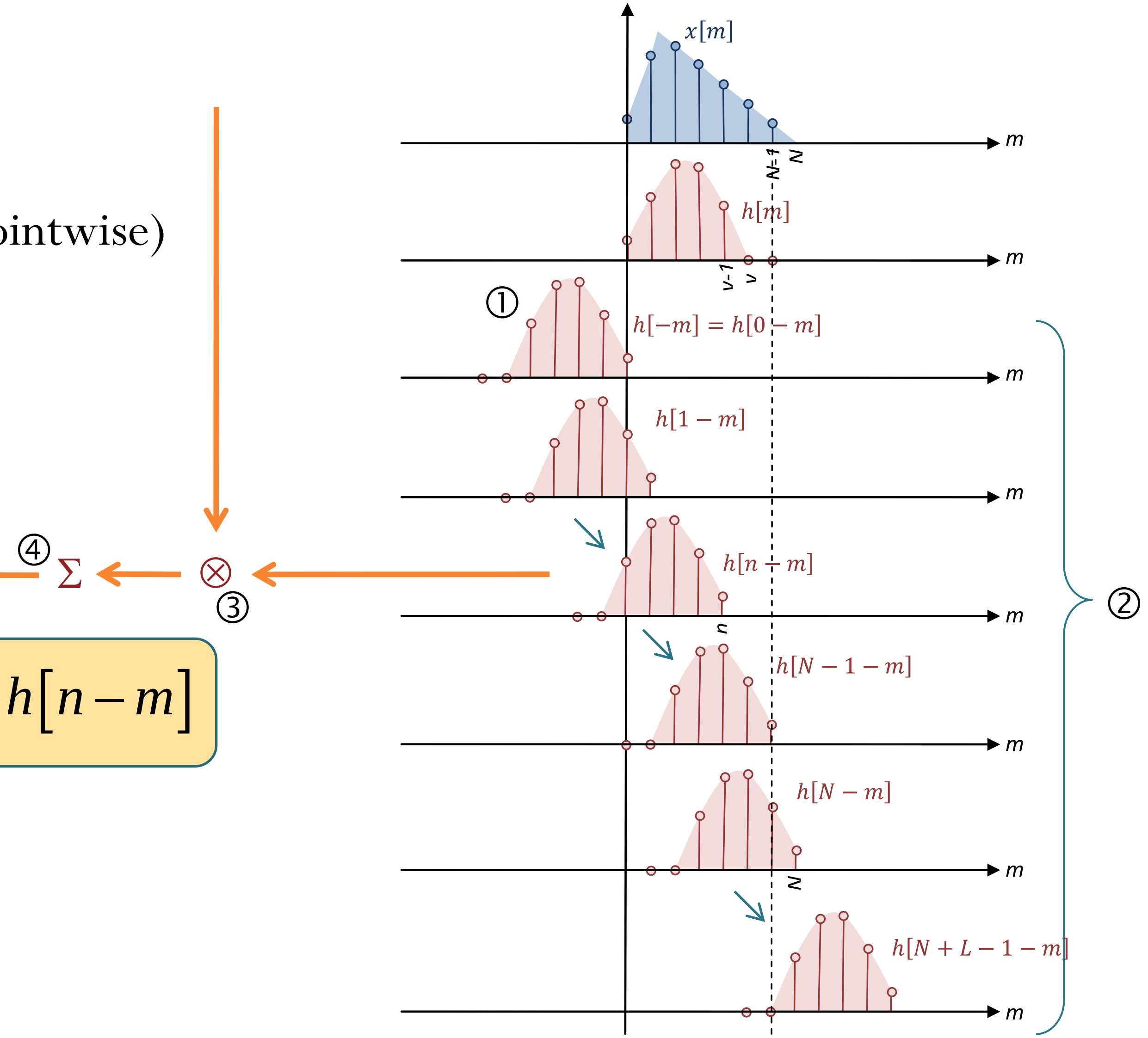
Next OFDM symbol

End of symbol is prepended to beginning
Guard interval still equals to T_{CP}

Using cyclic prefix:
OFDM symbol length: $T_{sym} + T_{CP}$
Efficiency: $T_{sym} / (T_{sym} + T_{CP})$

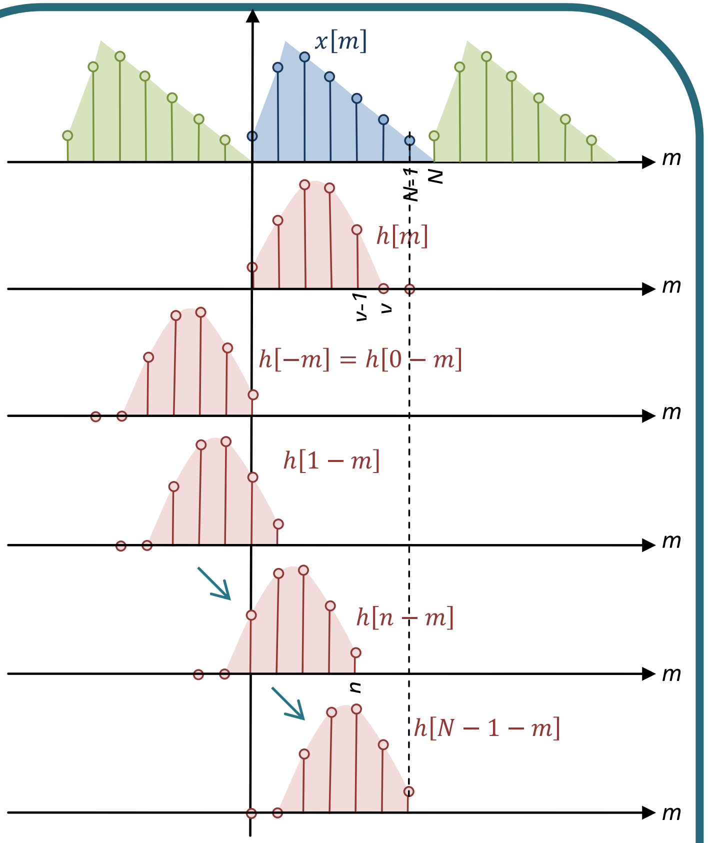
Recall: Convolution

① Flip
② Shift
③ Multiply (pointwise)
④ Add



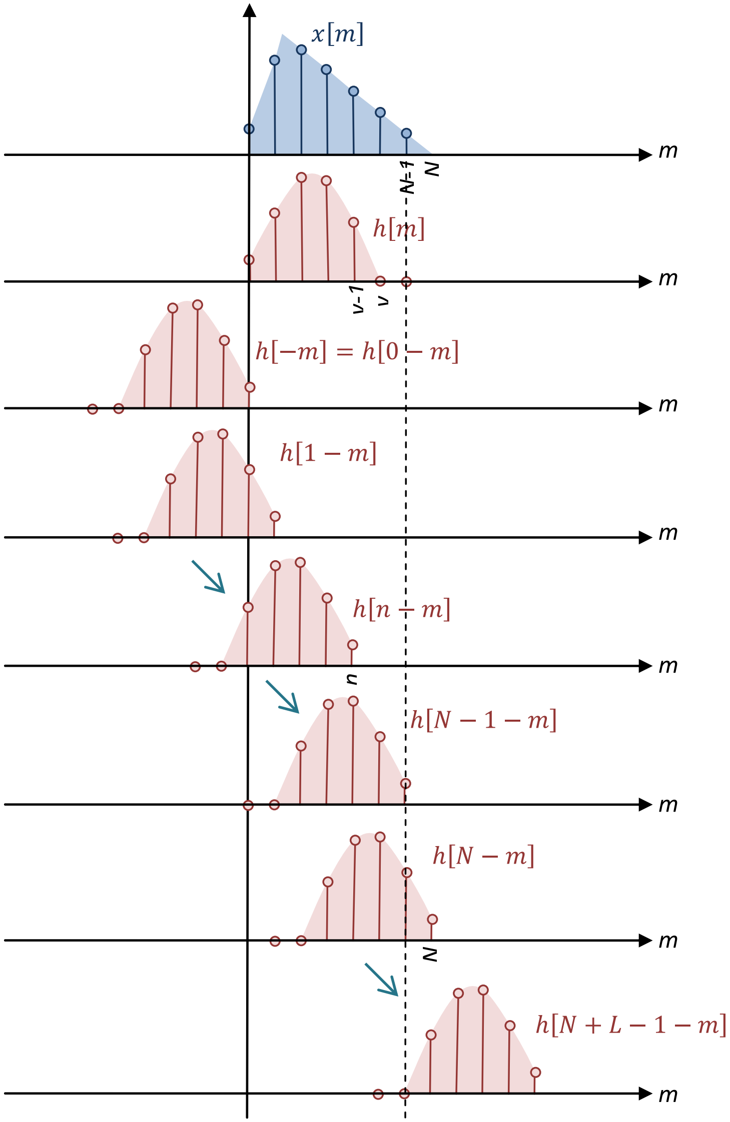
$$\{x*h\}[n]=\sum_{m}x[m]h[n-m]$$

Circular Convolution



Replicate x (now it looks periodic)
Then, perform the usual convolution
only on n = 0 to N-1

(Regular Convolution)



Circular Convolution: Examples 1

Find

$$[1 \quad 2 \quad 3] * [4 \quad 5 \quad 6]$$

$$[1 \quad 2 \quad 3] \circledast [4 \quad 5 \quad 6]$$

$$[1 \quad 2 \quad 3 \quad 0 \quad 0] \circledast [4 \quad 5 \quad 6 \quad 0 \quad 0]$$

Discussion

- ***Regular convolution*** of an N_1–point vector and an N_2–point vector gives (N_1+N_2-1)-point vector.
- ***Circular convolution*** is performed between two equal-length vectors. The results also has the same length.
- Circular convolution can be used to find the regular convolution by **zero-padding**.
  - Zero-pad the vectors so that their length is N_1+N_2-1.
  - Example:

$$[1\quad 2\quad 3\quad 0\quad 0] \circledast [4\quad 5\quad 6\quad 0\quad 0] = [1\quad 2\quad 3] * [4\quad 5\quad 6]$$

- In modern OFDM, we want to perform circular convolution via regular convolution.

Circular Convolution in Communication

- We want the receiver to obtain the circular convolution of the signal (channel input) and the channel.
- Q: Why?
- A:
  - **CTFT**: **convolution** in time domain corresponds to **multiplication** in frequency domain.
    - This fact does not hold for DFT.
  - **DFT**: circular **convolution** in (discrete) time domain corresponds to **multiplication** in (discrete) frequency domain.
    - We want to have multiplication in frequency domain.
    - So, we want circular convolution and not the regular convolution.
- Problem: Real channel does regular convolution.
- Solution: With **cyclic prefix**, regular convolution can be used to create circular convolution.

Example 2

$$[1\quad -2\quad 3\quad 1\quad 2]\circledast[3\quad 2\quad 1\quad 0\quad 0]=?$$

Let's look closer at how we carry out the circular convolution operation. Recall that we replicate the *x* and then perform the regular convolution (for *N* points)

Solution:

$$\begin{array}{cccccccccccccccc}
1 & -2 & 3 & 1 & 2 & 1 & -2 & 3 & 1 & 2 & 1 & -2 & 3 & 1 & 2 & \\
 & 0 & 0 & 1 & 2 & 3 & & & & & & & & & & 1\times1+2\times2+1\times3=1+4+3=8 \\
 & & 0 & 0 & 1 & 2 & 3 & & & & & & & & & 2\times1+1\times2+(-2)\times3=2+2-6=-2 \\
 & & & 0 & 0 & 1 & 2 & 3 & & & & & & & & 1\times1+(-2)\times2+3\times3=1-4+9=6 \\
 & & & & 0 & 0 & 1 & 2 & 3 & & & & & & & (-2)\times1+3\times2+1\times3=-2+6+3=7 \\
 & & & & & 0 & 0 & 1 & 2 & 3 & & & & & & 3\times1+1\times2+2\times3=3+2+6=11
\end{array}$$

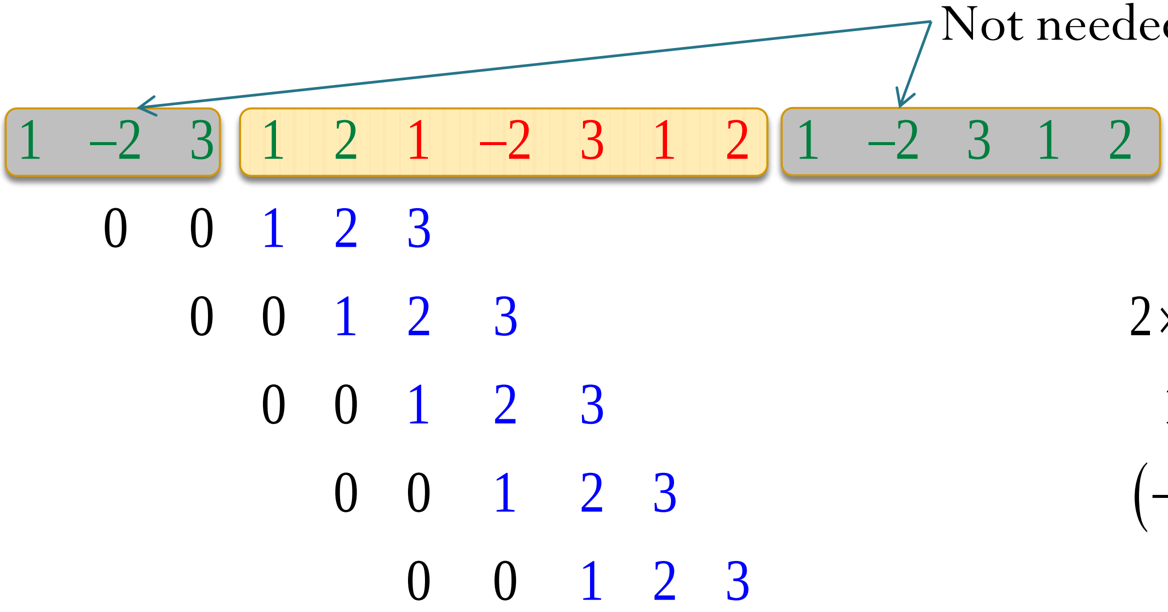
$$[1\quad -2\quad 3\quad 1\quad 2]\circledast[3\quad 2\quad 1\quad 0\quad 0]=\underbrace{[8\quad -2\quad 6\quad 7\quad 11]}$$

Goal: Get these numbers using regular convolution

Example 2

$$[1\quad -2\quad 3\quad 1\quad 2]\circledast[3\quad 2\quad 1\quad 0\quad 0]=?$$

Observation: We don't need to replicate the x indefinitely. Furthermore, when h is shorter than x, we need only a part of one replica.



Not needed in the calculation

$$1\times1+2\times2+1\times3=1+4+3=8$$

$$2\times1+1\times2+(-2)\times3=2+2-6=-2$$

$$1\times1+(-2)\times2+3\times3=1-4+9=6$$

$$(-2)\times1+3\times2+1\times3=-2+6+3=7$$

$$3\times1+1\times2+2\times3=3+2+6=11$$

$$[1\quad -2\quad 3\quad 1\quad 2]\circledast[3\quad 2\quad 1\quad 0\quad 0]=[8\quad -2\quad 6\quad 7\quad 11]$$

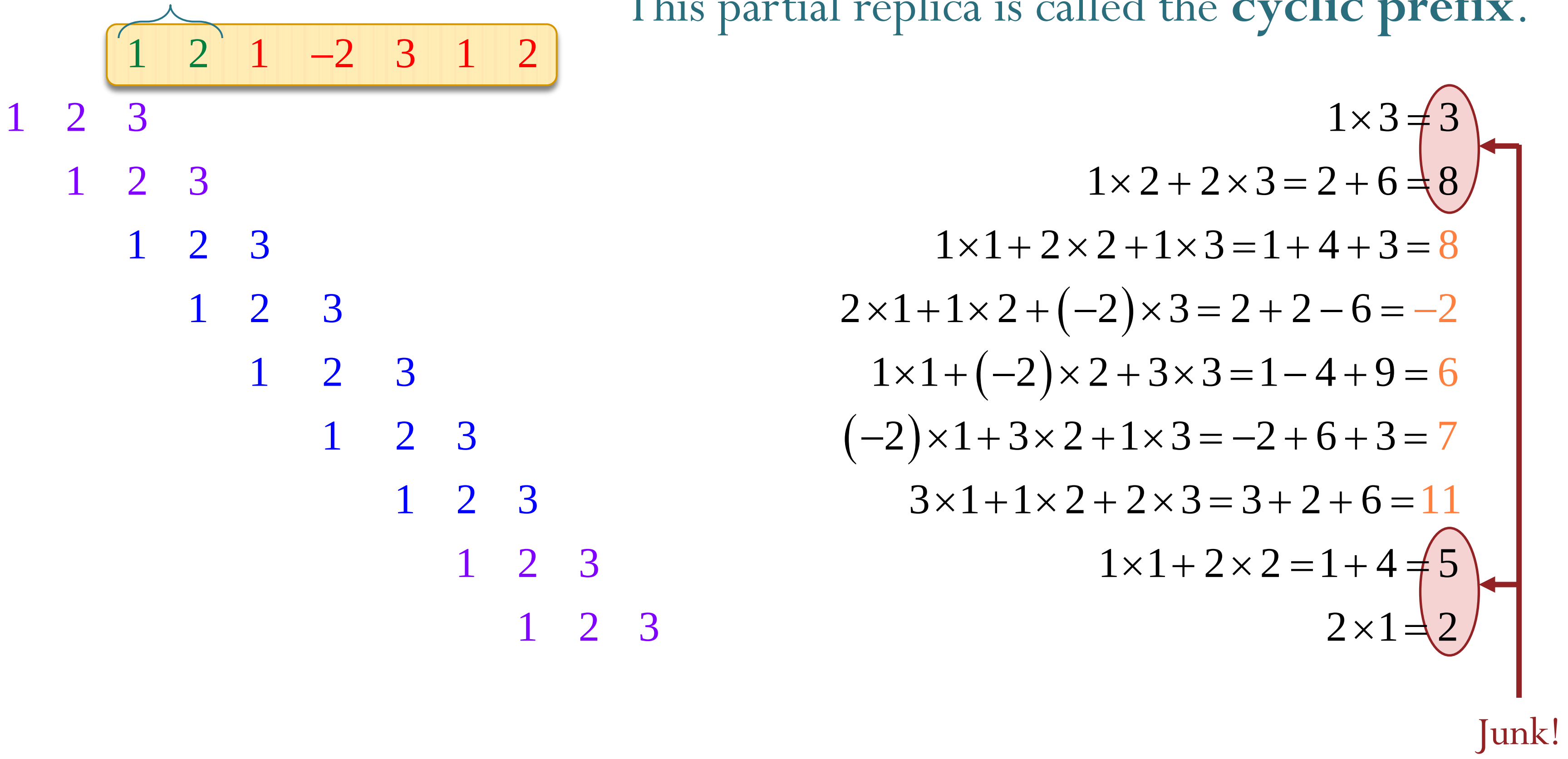
Try this: use only the necessary part of the replica and then convolute (regular convolution) with the channel.

Example 2

$$[\underbrace{1\quad 2}\quad 1\quad -2\quad 3\quad 1\quad 2]*[3\quad 2\quad 1]=?$$

Copy the **last ν samples** of the symbols at the **beginning** of the symbol.

This partial replica is called the **cyclic prefix**.



$$\begin{array}{ccccccccccc r}
 & & \multicolumn{2}{c}{\overbrace{1\quad 2}} & 1 & -2 & 3 & 1 & 2 & & & \\
1 & 2 & 3 & & & & & & & & & 1\times 3=3 \\
 & 1 & 2 & 3 & & & & & & & & 1\times 2+2\times 3=2+6=8 \\
 & & 1 & 2 & 3 & & & & & & & 1\times 1+2\times 2+1\times 3=1+4+3=8 \\
 & & & 1 & 2 & 3 & & & & & & 2\times 1+1\times 2+(-2)\times 3=2+2-6=-2 \\
 & & & & 1 & 2 & 3 & & & & & 1\times 1+(-2)\times 2+3\times 3=1-4+9=6 \\
 & & & & & 1 & 2 & 3 & & & & (-2)\times 1+3\times 2+1\times 3=-2+6+3=7 \\
 & & & & & & 1 & 2 & 3 & & & 3\times 1+1\times 2+2\times 3=3+2+6=11 \\
 & & & & & & & 1 & 2 & 3 & & 1\times 1+2\times 2=1+4=5 \\
 & & & & & & & & 1 & 2 & 3 & 2\times 1=2
\end{array}$$

Junk!

Example 2

- We now know that

$$\underbrace{[1\quad 2}_{\text{Cyclic Prefix}}\quad 1\quad -2\quad 3\quad 1\quad 2] * [3\quad 2\quad 1] = [3\quad 8\quad \underbrace{8\quad -2\quad 6\quad 7\quad 11}_{[1\quad -2\quad 3\quad 1\quad 2] \circledast [3\quad 2\quad 1\quad 0\quad 0]}\quad 5\quad 2]$$

- Similarly, you may check that

$$\underbrace{[-2\quad 1}_{\text{Cyclic Prefix}}\quad 2\quad 1\quad -3\quad -2\quad 1] * [3\quad 2\quad 1] = [-6\quad -1\quad \underbrace{6\quad 8\quad -5\quad -11\quad -4}_{[2\quad 1\quad -3\quad -2\quad 1] \circledast [3\quad 2\quad 1\quad 0\quad 0]}\quad 0\quad 1]$$

Example 3

- We know, from Example 2, that

```
[ 1 2 1 -2  3  1 2] * [3 2 1] = [ 3  8 8 -2  6   7 11 5 2]
```

And that

```
[-2 1 2  1 -3 -2 1] * [3 2 1] = [-6 -1 6  8 -5 -11 -4 0 1]
```

- Check that

```
  [ 1 2 1 -2 3  1  2  0  0 0 0  0   0  0] * [3 2 1]
= [ 3 8 8 -2 6  7 11  5  2 0 0  0   0  0 0 0]
```

and

```
  [ 0 0 0  0 0  0  0 -2  1 2 1 -3  -2  1] * [3 2 1]
= [ 0 0 0  0 0  0  0 -6 -1 6 8 -5 -11 -4 0 1]
```

Example 4

- We know that

```
[ 1 2 1 -2  3  1 2] * [3 2 1] = [ 3  8 8 -2  6   7 11 5 2]
[-2 1 2  1 -3 -2 1] * [3 2 1] = [-6 -1 6  8 -5 -11 -4 0 1]
```

- Using Example 3, we have

```
    [ 1 2 1 -2 3  1  2 -2  1 2 1 -3  -2  1] * [3 2 1]

= ( [ 1 2 1 -2 3  1  2  0  0 0 0  0   0  0] )
   +[ 0 0 0  0 0  0  0 -2  1 2 1 -3  -2  1] )  * [3 2 1]

=   [ 3 8 8 -2 6  7 11  5  2 0 0  0   0  0 0 0]
   +[ 0 0 0  0 0  0  0 -6 -1 6 8 -5 -11 -4 0 1]

=   [ 3 8 8 -2 6  7 11 -1  1 6 8 -5 -11 -4 0 1]
```

Putting results together…

- Suppose $\underline{x}^{(1)} = [1\ \ -2\ \ 3\ \ 1\ \ 2]$ and $\underline{x}^{(2)} = [2\ \ 1\ \ -3\ \ -2\ \ 1]$
- Suppose $\underline{h} = [3\ \ 2\ \ 1]$
- At the receiver, we want to get
  - $[1\ \ -2\ \ 3\ \ 1\ \ 2] \circledast [3\ \ 2\ \ 1\ \ 0\ \ 0] = [8\ \ -2\ \ 6\ \ 7\ \ 11]$
  - $[2\ \ 1\ \ -3\ \ -2\ \ 1] \circledast [3\ \ 2\ \ 1\ \ 0\ \ 0] = [6\ \ 8\ \ -5\ \ -11\ \ -4]$
- We transmit $[\underbrace{1\ \ 2}_{\text{Cyclic prefix}}\ \ 1\ \ -2\ \ 3\ \ 1\ \ 2\ \ \underbrace{-2\ \ 1}_{\text{Cyclic prefix}}\ \ 2\ \ 1\ \ -3\ \ -2\ \ 1]$.
- At the receiver, we get

$$[\ 1\ \ 2\ \ 1\ \ -2\ \ 3\ \ 1\ \ 2\ \ -2\ \ 1\ \ 2\ \ 1\ \ -3\ \ -2\ \ 1] * [3\ \ 2\ \ 1]$$

$$= [\underbrace{3\ \ 8}\ \ 8\ \ -2\ \ 6\ \ 7\ \ 11\ \ \underbrace{-1\ \ 1}\ \ 6\ \ 8\ \ -5\ \ -11\ \ -4\ \ \underbrace{0\ \ 1}]$$

Junk! To be thrown away by the receiver.

Circular Convolution: Key Properties

- Consider an N-point signal $x[n]$
- **Cyclic Prefix (CP) insertion**: If $x[n]$ is extended by copying the last ν samples of the symbols at the beginning of the symbol:

$$\hat{x}[n] = \begin{cases} x[n], & 0 \le n \le N-1 \\ x[n+N], & -v \le n \le -1 \end{cases}$$

- Key Property 1:

$$\{h \circledast x\}[n] = (h * \hat{x})[n] \text{ for } 0 \le n \le N-1$$

- Key Property 2:

$$\{h \circledast x\}[n] \xrightarrow{\text{FFT}} \boxed{H_k X_k}$$

OFDM with CP for Channel w/ Memory

- We want to send N samples S_0, S_1, …, S_{N-1} across noisy channel with memory.
- First apply IFFT: $S_k \xrightarrow{\text{IFFT}} s[n]$
- Then, add cyclic prefix

$$\hat{s} = \left[s[N-v], \ldots, s[N-1], s[0], \ldots, s[N-1] \right]$$

- This is inputted to the channel.
- The output is

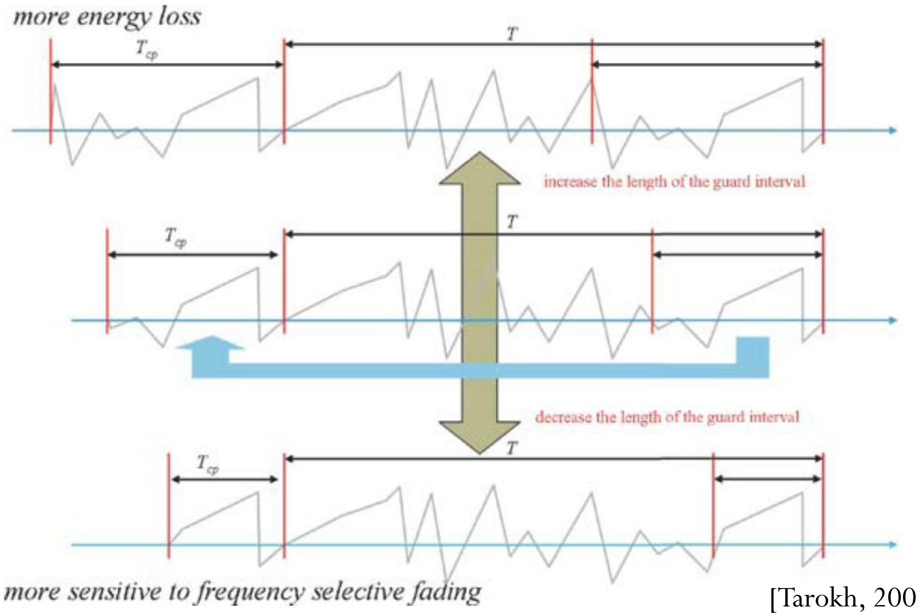
$$y[n] = \left[p[N-v], \ldots, p[N-1], r[0], \ldots, r[N-1] \right]$$

- Remove cyclic prefix to get $r[n] = h[n] \circledast s[n] + w[n]$
- Then apply FFT: $r[n] \xrightarrow{\text{FFT}} R_k$
- By circular convolution property of DFT, $R_k = H_k S_k + W_k$

No ICI!

OFDM System Design: CP

- A good ratio between the CP interval and symbol duration should be found, so that all multipaths are resolved and not significant amount of energy is lost due to CP.
- As a thumb rule, the CP interval must be two to four times larger than the root mean square (RMS) delay spread.



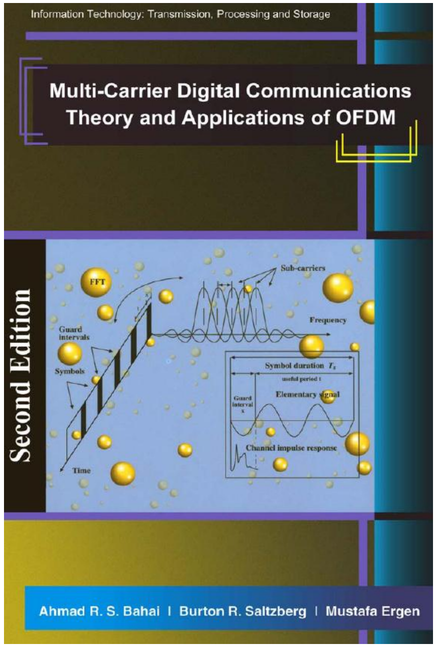
[Tarokh, 2009, Fig 2.9]

Summary

- The CP at the beginning of each block has two main functions.
- As guard interval, it prevents contamination of a block by ISI from the previous block.
- It makes the received block appear to be periodic of period N.
  - Turn regular convolution into circular convolution
  - Point-wise multiplication in the frequency domain

Reference

- A. Bahai, B. R. Saltzberg, and M. Ergen, ***Multi-Carrier Digital Communications:Theory and Applications of OFDM***, 2nd ed., New York: Springer Verlag, 2004.



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OFDM

5.5 Remarks about OFDM

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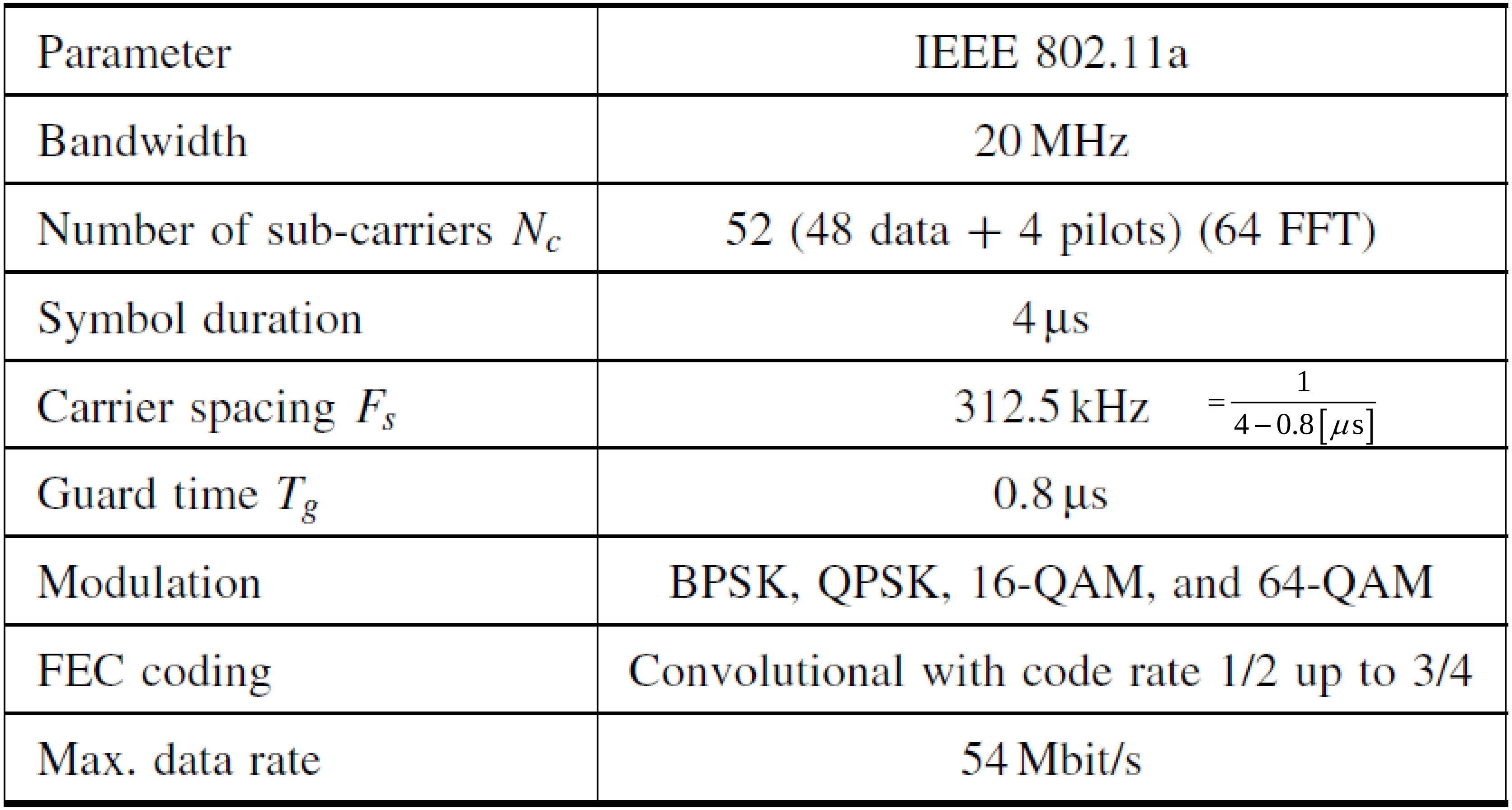
Summary: OFDM Advantages

- For a given channel delay spread, the **implementation complexity** is much lower than that of a conventional single carrier (SC) system with *time domain equalizer*.
- **Spectral efficiency** is high since it uses overlapping orthogonal subcarriers in the frequency domain.
- Modulation and demodulation are implemented using inverse discrete Fourier transform (IDFT) and discrete Fourier transform (DFT), respectively, and fast Fourier transform (FFT) algorithms can be applied to make the overall system **efficient** (computationally).
- **Capacity** can be significantly increased by **adapting the data rate per subcarrier** according to the signal-to-noise ratio (**SNR**) of the individual subcarrier.

Example: 802.11a



Parameter	IEEE 802.11a
Bandwidth	20 MHz
Number of sub-carriers N_c	52 (48 data + 4 pilots) (64 FFT)
Symbol duration	4 µs
Carrier spacing F_s	312.5 kHz $= \frac{1}{4-0.8[\mu s]}$
Guard time T_g	0.8 µs
Modulation	BPSK, QPSK, 16-QAM, and 64-QAM
FEC coding	Convolutional with code rate 1/2 up to 3/4
Max. data rate	54 Mbit/s

OFDM Drawbacks

- **High peak-to-average power ratio (PAPR)**
  - The transmitted signal is a superposition of all the subcarriers with different carrier frequencies and high amplitude peaks occur because of the superposition.
- High sensitivity to **frequency offset**:
  - When there are frequency offsets in the subcarriers, the orthogonality among the subcarriers breaks and it causes intercarrier interference (ICI).
- A need for an adaptive or coded scheme to overcome **spectral nulls** in the channel
  - In the presence of a null in the channel, there is no way to recover the data of the subcarriers that are affected by the null unless we use rate adaptation or a coding scheme.

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OFDM

5.6 **OFDM-Based Multiple Access Techniques**

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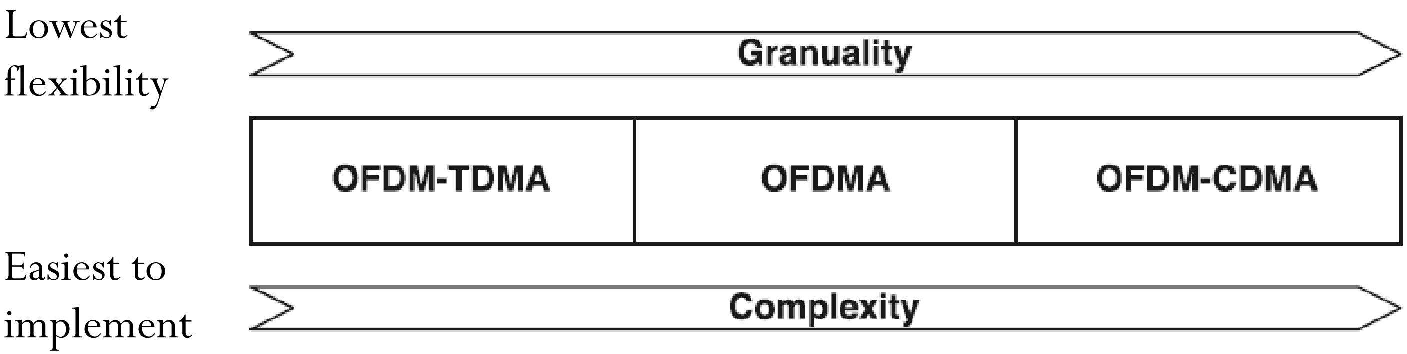
OFDM-based Multiple Access

1. OFDMA
2. OFDM-TDMA
3. Multi-Carrier CDMA (OFDM-CDMA)



Lowest flexibility

Granuality

OFDM-TDMA

OFDMA

OFDM-CDMA

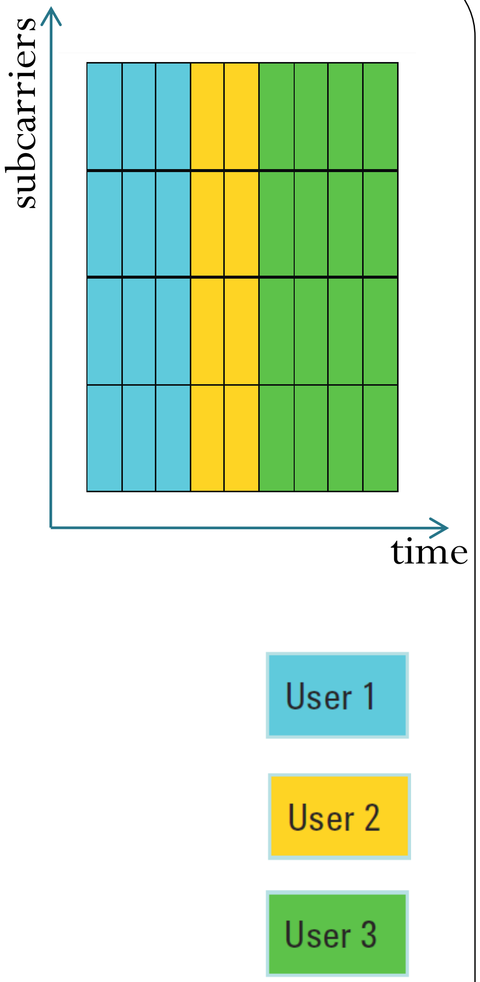
Easiest to implement

Complexity

[Tarokh, 2009, Section 2.9, Fig 2.10]

OFDM-TDMA

- Users are separated via **time slots**.
- A particular user uses **all** sub-carriers within the predetermined TDMA time slot.
- Example: 802.11
  - Each user uses OFDM modulation and gets **transmission right** through the MAC layer channel access mechanism.



OFDM-TDMA (2)

- **Advantage**:
  - MS can **reduce** its **power consumption**
    - Process only OFDM symbols which are dedicated to it.
- **Disadvantage**:
  - Allocate the whole bandwidth to a single user
    - A reaction to different subcarrier attenuations could consist of leaving out highly distorted subcarriers

OFDMA

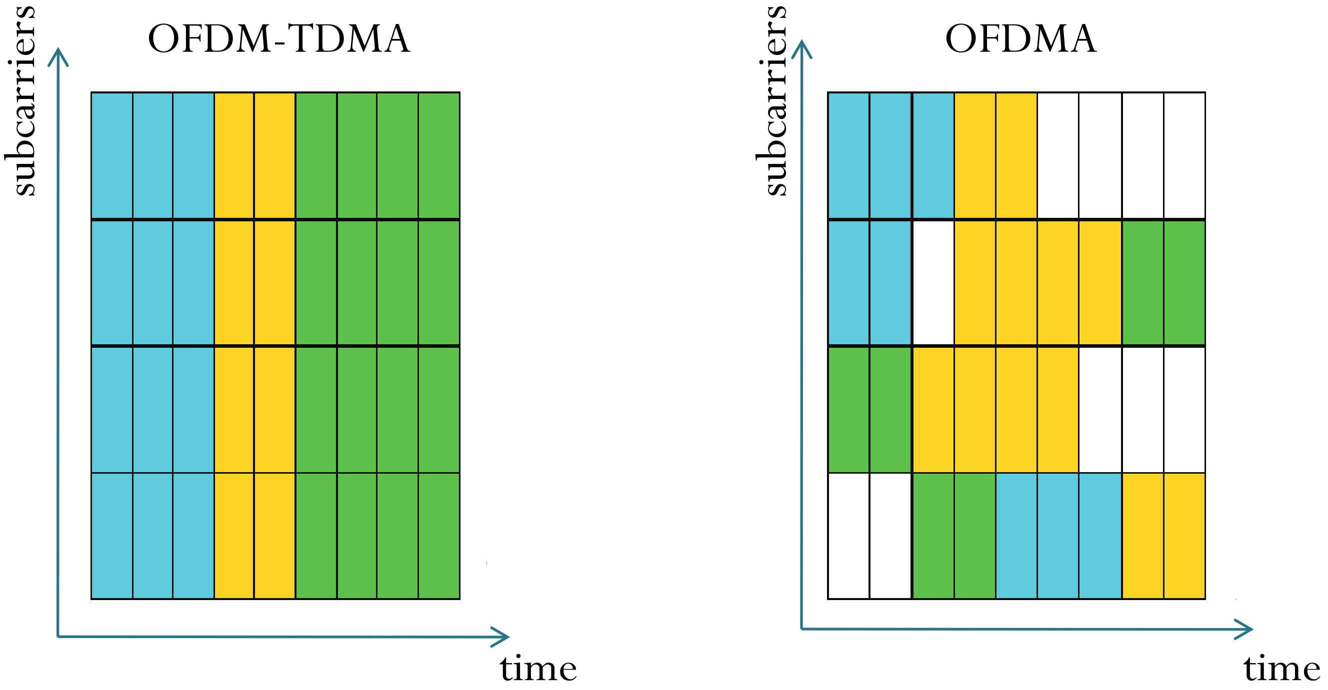
- Available subcarriers are distributed among all the users for transmission at any time instant.
- The fact that each user experiences a different radio channel can be exploited by allocating only “good” subcarriers with high SNR to each user.
  - *Recall*: For OFDM system, based on the **subchannel condition**, **different** baseband **modulation schemes** can be used for the individual subchannels
- The number of subchannels for a specific user can be varied, according to the required data rate.

OFDM-TDMA vs. OFDMA

User 1

User 2

User 3



OFDMA Block Diagram